CHRISTIAN EMINENT COLLEGE, INDORE

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E-Content

On

" Vectors 1 "

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Mathematics

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Session

Vectors -1

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Session Objectives

- Scalar or Dot Product
- Geometrical Interpretation: Projection of a Vector
- Properties of Scalar Product
- Scalar Product in Terms of Components
- Components of a Vector $\vec{b}$ along and perpendicular to $\vec{a}$
- Geometrical Problems
- Application: Work Done by a Force
- Class Exercise

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Scalar or Dot Product

Given two non-zero vectors $\vec{a}$ and $\vec{b}$ inclined at an angle θ, the scalar product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\,.\,\vec{b}$, is defined as

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \qquad \text{where } 0 \le \theta \le \pi$$

Note: 1. The scalar product of two vectors is always a scalar quantity.

2. If $\theta = \dfrac{\pi}{2}$, then $\vec{a}.\vec{b} = 0$

3. If $\vec{a}$ or $\vec{b}$ or both is a zero vector, then θ is not defined as $\vec{0}$ has no direction and $\vec{a}.\vec{b} = 0$.

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Angle Between Two Vectors in Terms of Scalar Product

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

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Orthonormal Vector Triad

Let $\hat{i}, \hat{j}$ and $\hat{k}$ be unit vectors along three mutually perpendicular coordinate axis, x-axis, y-axis and z-axis respectively.

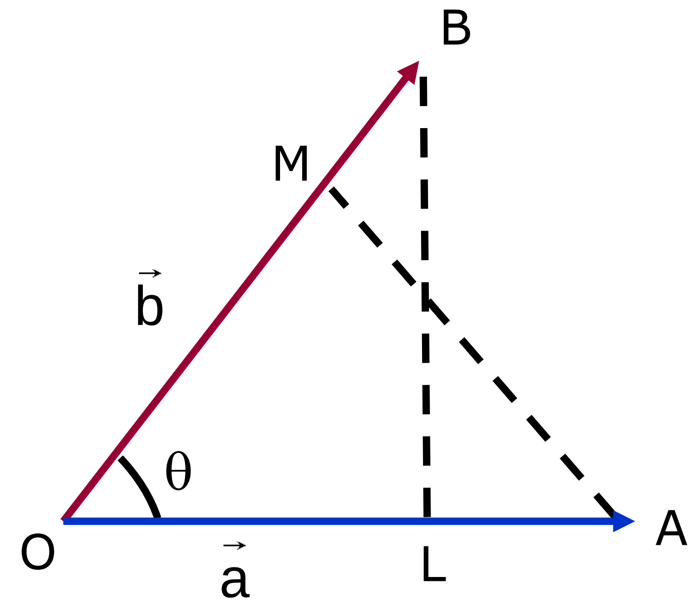
$$\hat{i}.\hat{i} = |\hat{i}||\hat{i}|\cos 0 = 1$$

Similarly, $\hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

Also, $\hat{i}.\hat{j} = |\hat{i}||\hat{j}|\cos 90° = 1 \times 1 \times 0 = 0$

Similarly, $\hat{i}.\hat{k} = \hat{j}.\hat{k} = \hat{j}.\hat{i} = \hat{k}.\hat{i} = \hat{k}.\hat{j} = 0.$

Geometrical Interpretation of Scalar Product



$OL = OB\cos\theta$ and $OM = OA\cos\theta$

OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

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Projection of a Vector

$$\vec{a}\,.\,\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$$

$$\Rightarrow \vec{a}\,.\,\vec{b} = \left|\vec{a}\right|(OB\cos\theta) \Rightarrow \vec{a}\,.\,\vec{b} = \left|\vec{a}\right|(OL)$$

$$\Rightarrow OL = \frac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|}$$

Length of projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|} = \dfrac{\vec{a}}{\left|\vec{a}\right|}.\vec{b} = \hat{a}\,.\,\vec{b}$

Lenth of projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\,.\,\vec{b}}{\left|\vec{b}\right|} = \vec{a}\,.\,\dfrac{\vec{b}}{\left|\vec{b}\right|} = \vec{a}\,.\,\hat{b}$

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Properties of Scalar Product

1. Scalar product of two vectors is commutative

i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

2. Scalar product of vectors is distributive

(i) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$ (Left distributive)

(ii) $(\vec{b}+\vec{c}).\vec{a} = \vec{b}.\vec{a}+\vec{c}.\vec{a}$ (Right distributive)

3. Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors. Then $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$.

Properties of Scalar Product (Cont.)

4. For any vector $\vec{a}$, $\vec{a}.\vec{a} = |\vec{a}|^2$.

5. If m is a scalar and $\vec{a}$ and $\vec{b}$ be any two vectors, then

$$(m\,\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.(m\,\vec{b})$$

6. If $\vec{a}$ and $\vec{b}$ are two vectors, then

(i) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$

(ii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

(iii) $(\vec{a}+\vec{b})(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

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Scalar Product in Terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$$\vec{a}\,.\,\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$$

$$\therefore \theta = \cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}\right)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}\sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}}\right)$$

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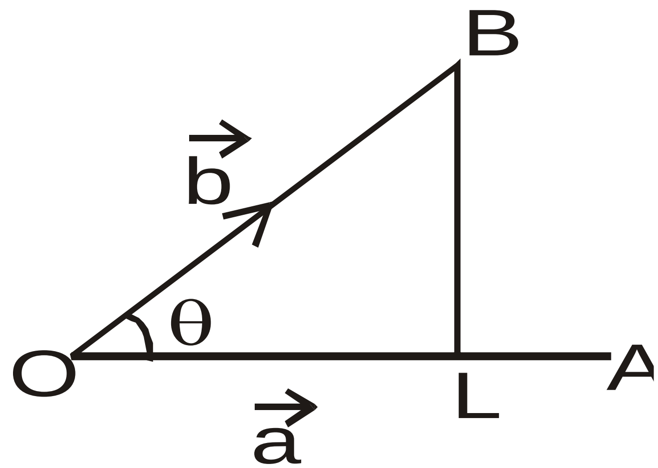
Components of a Vector $\vec{b}$ along and perpendicular to $\vec{a}$

$$\overrightarrow{OL} = (OL)\hat{a} = (OB\cos\theta)\hat{a}$$

$$= \left(|\vec{b}|\cos\theta\right)\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\left(|\vec{a}||\vec{b}|\cos\theta\right)\vec{a}}{|\vec{a}|^2} = \frac{\left(\vec{a}\,.\,\vec{b}\right)\vec{a}}{|\vec{a}|^2}$$

$$\overrightarrow{LB} = \overrightarrow{OB} - \overrightarrow{OL} = \vec{b} - \frac{\left(\vec{a}\,.\,\vec{b}\right)\vec{a}}{|\vec{a}|^2}$$



Example -1

If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$, then find $\left(\vec{a} + 3\vec{b}\right).\left(2\vec{a} - \vec{b}\right)$.

Solution: We have $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$$\therefore \left(\vec{a} + 3\vec{b}\right) = \left(\hat{i} + \hat{j} + 2\hat{k}\right) + 3\left(3\hat{i} + 2\hat{j} - \hat{k}\right) = 10\hat{i} + 7\hat{j} - \hat{k}$$

$$\text{and } \left(2\vec{a} - \vec{b}\right) = 2\left(\hat{i} + \hat{j} + 2\hat{k}\right) - \left(3\hat{i} + 2\hat{j} - \hat{k}\right) = -\hat{i} + 5\hat{k}$$

$$\therefore \left(\vec{a} + 3\vec{b}\right).\left(2\vec{a} - \vec{b}\right) = \left(10\hat{i} + 7\hat{j} - \hat{k}\right).\left(-\hat{i} + 0\hat{j} + 5\hat{k}\right)$$

$$= -10 + 0 - 5 = -15$$

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Example –2

Find the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other.

Solution: The vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, if

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow \left(2\hat{i} + \lambda\hat{j} + \hat{k}\right) . \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = 0$$

$$\Rightarrow 2 - 2\lambda + 3 = 0 \Rightarrow \lambda = \frac{5}{2}$$

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Example –3

Dot product of a vector with vectors $\hat{i} - \hat{j}$, $\hat{i} + \hat{k}$ and $\hat{i} + \hat{j} - 2\hat{k}$ are respectively 1, - 6 and 3. Find the vector.

Solution: Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required vector.

$$\vec{r}\,.\,\vec{a} = 1 \Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} - \hat{j}\right) = 1 \Rightarrow x - y = 1 \qquad ...(i)$$

$$\vec{r}\,.\,\vec{b} = -6 \Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + \hat{k}\right) = -6 \Rightarrow x + z = -6 \qquad ...(ii)$$

Solution Cont.

$$\vec{r}\,.\,\vec{c}=3 \Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}+\hat{j}-2\,\hat{k}\right)=3 \Rightarrow x+y-2z=3 \qquad ...(iii)$$

Solving (i), (ii) and (iii), we get

$x = -2$, $y = -3$ and $z = -4$

Hence, the vector is $-2\hat{i}-3\hat{j}-4\hat{k}$.

Example –4

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$, find $(\vec{a}\,.\,\vec{b})+(\vec{b}\,.\,\vec{c})+(\vec{c}\,.\,\vec{a})$.

Solution: $\because \vec{a}+\vec{b}+\vec{c}=0 \Rightarrow (\vec{a}+\vec{b}+\vec{c})^2=0$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$$

$$\Rightarrow 9+16+25+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$$

$$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=-25$$

Example –5

Find the angle between the vectors $2\hat{i} - 3\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - 2\hat{k}$.

Solution: Let $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and θ be the angle between them. Then,

$$\cos\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\vec{a}\,.\,\vec{b} = \left(2\hat{i} - 3\hat{j} + \hat{k}\right).\left(\hat{i} + \hat{j} - 2\hat{k}\right) = 2 - 3 - 2 = -3$$

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Solution Cont.

$$\left|\vec{a}\right| = \sqrt{(2)^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\left|\vec{b}\right| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$\therefore \cos\theta = \frac{-3}{\sqrt{14}\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{-3}{\sqrt{84}}\right)$$

Example –6

Find the length of projection of $\vec{a} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k}$.

Solution : Given vectors are $\vec{a} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k}$.

The length of projrction of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\,.\,\vec{b}}{\left|\vec{b}\right|}$

$$= \frac{\left(3\hat{i} - 2\hat{j} + 4\hat{k}\right).\left(2\hat{i} - 4\hat{j} + \hat{k}\right)}{\sqrt{(2)^2 + (-4)^2 + (1)^2}}$$

$$= \frac{6 + 8 + 4}{\sqrt{4 + 16 + 1}} = \frac{18}{\sqrt{21}}$$

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Example –7

Find the value of p so that the two vectors

$3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}+p\hat{j}+3\hat{k}$ are

(i) parallel (ii) perpendicular to each other

Solution: (i) Vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}+p\hat{j}+3\hat{k}$ are parallel iff

$$\frac{3}{1}=\frac{2}{p}=\frac{9}{3}\Rightarrow p=\frac{2}{3}$$

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Solution Cont.

(ii) Vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}+p\hat{j}+3\hat{k}$ are perpendicular iff

$$\left(3\hat{i}+2\hat{j}+9\hat{k}\right).\left(\hat{i}+p\hat{j}+3\hat{k}\right)=0$$

$$\Rightarrow 3+2p+27=0$$

$$\Rightarrow p=-15$$

Geometrical Problems
Example -8

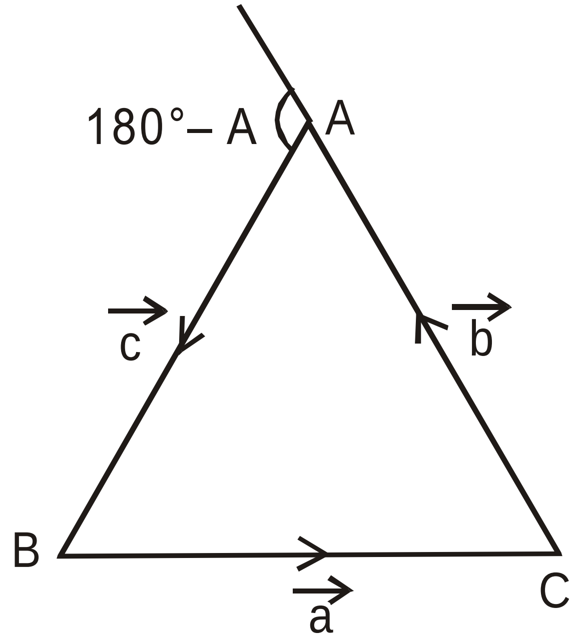
Prove the cosine formula for a triangle, i.e. if a, b and c are the lengths of the opposite sides respectively to the angles A, B and C of a triangle ABC, then

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Solution



By the triangle law of addition, we have

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$\Rightarrow \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

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Solution Cont.

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$$

$$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$$

$$\Rightarrow \left(\vec{b} + \vec{c}\right).\left(\vec{b} + \vec{c}\right) = \left(-\vec{a}\right).\left(-\vec{a}\right)$$

$$\Rightarrow \left|\vec{b} + \vec{c}\right|^2 = \left|\vec{a}\right|^2$$

$$\Rightarrow \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\vec{b}.\vec{c} = \left|\vec{a}\right|^2$$

Solution Cont.

$$\Rightarrow \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\left|\vec{b}\right|.\left|\vec{c}\right|\cos(180° - A) = \left|\vec{a}\right|^2$$

$$\Rightarrow b^2 + c^2 - 2bc\cos A = a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similarly, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example -9

Show that the diagonals of a rhombus bisect each other at right-angles.

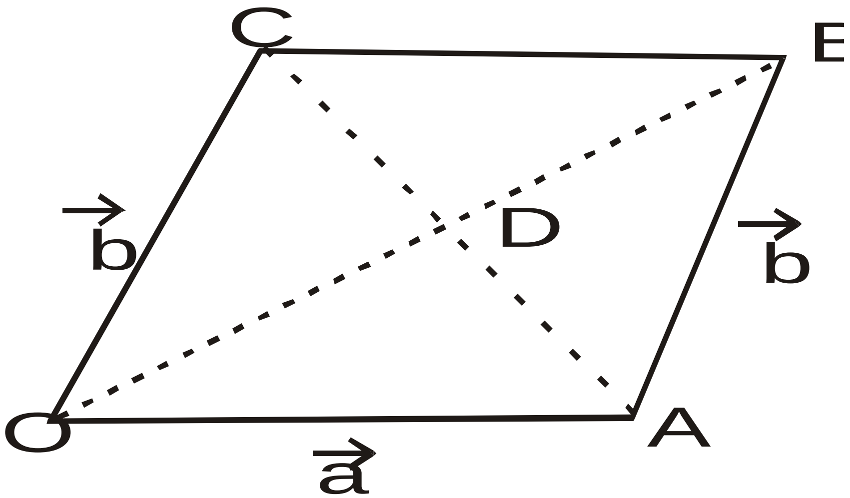
Solution:

$$\overrightarrow{OB} = \vec{a} + \vec{b} \text{ and } \overrightarrow{AC} = \vec{b} - \vec{a}$$

$$\left|\overrightarrow{OA}\right| = \left|\overrightarrow{AB}\right| \Rightarrow \left|\vec{a}\right| = \left|\vec{b}\right| \Rightarrow a = b$$

$$\left(\vec{a} + \vec{b}\right).\left(\vec{b} - \vec{a}\right)$$

$$= \left(\vec{b} + \vec{a}\right).\left(\vec{b} - \vec{a}\right)$$



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Solution Cont.

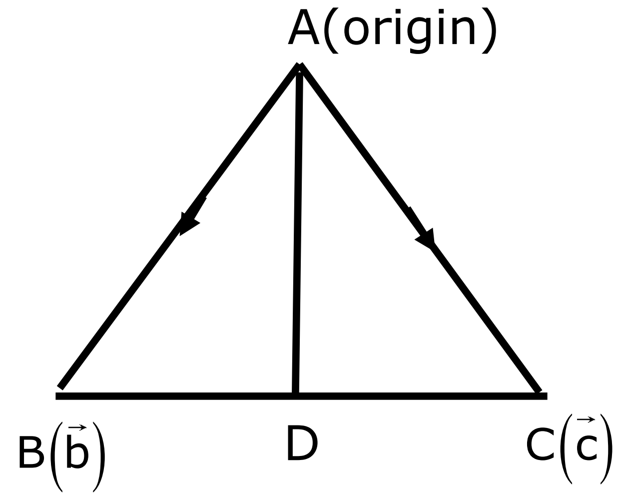
$$= b^2 - a^2 = 0$$

$$\overrightarrow{OB} \perp \overrightarrow{AC}$$

Hence, diagonals of a rhombus are at right angles.

Example -10

Prove using vectors: The median to the base of an isosceles triangle is perpendicular to the base.



Solution: Let ABC be an isosceles triangle in which AB = AC.

$$\Rightarrow |\vec{b}| = |\vec{c}|$$

$$\text{Position vector of D} = \frac{\vec{b}+\vec{c}}{2}$$

$$\overrightarrow{AD} = \frac{\vec{b}+\vec{c}}{2} - \vec{0} = \frac{1}{2}(\vec{b}+\vec{c})$$

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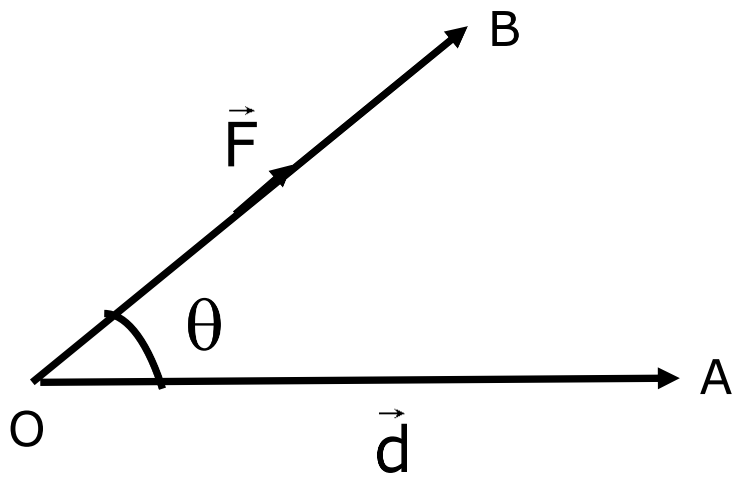
Solution Cont.

$$\overrightarrow{BC} = \vec{c} - \vec{b}$$

$$\overrightarrow{AD} \, . \, \overrightarrow{BC} = \frac{1}{2}\left(\vec{b} + \vec{c}\right).\left(\vec{c} - \vec{b}\right) = \frac{1}{2}\left(\left|\vec{c}\right|^2 - \left|\vec{b}\right|^2\right) = 0$$

$$\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow AD \perp BC$$

Application: Work Done by a Force



Let $\overrightarrow{OA}$ be the displacement. Then the component of $\overrightarrow{OA}$ in the direction of the force $\vec{F}$ is $\left|\overrightarrow{OA}\right|\cos\theta$.

$$\therefore \text{Work done} = \left|\vec{F}\right|\left|\overrightarrow{OA}\right|\cos\theta = \vec{F}.\overrightarrow{OA} = \vec{F}.\vec{d}$$

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Example -11

Find the work done by a force $\vec{F} = 3\hat{i} + 2\hat{j} + \hat{k}$ acting on a particle, if the particle is displaced from the point with position vector $2\hat{i} + 3\hat{j} + 2\hat{k}$ to the point with vector $3\hat{i} + 3\hat{j} + 3\hat{k}$.

Solution: We have $\vec{F} = 3\hat{i} + 2\hat{j} + \hat{k}$

Let A and B be the points with position vectors $2\hat{i} + 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ respectively. Then the displacement $\vec{d}$ is given by

$$\vec{d} = \overrightarrow{AB} = \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) = \hat{i} + 0\hat{j} + \hat{k}$$

$$\therefore \text{Work done} = \vec{F}.\vec{d} = \left(3\hat{i} + 2\hat{j} + \hat{k}\right).\left(\hat{i} + 0\hat{j} + \hat{k}\right)$$

$$= 3 + 0 + 1 = 4 \text{ sq. units.}$$

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Example -12

Forces of magnitudes 5, 3, 1 units acting in the directions $6\hat{i}+2\hat{j}+3\hat{k}$, $3\hat{i}-2\hat{j}+6\hat{k}$, $2\hat{i}-3\hat{j}-6\hat{k}$ respectively act on a particle which is displaced from the point $(2,-1,-3)$ to $(5,-1,1)$.

Solution :

$$\text{Total force } \vec{F} = 5\frac{\left(6\hat{i}+2\hat{j}+3\hat{k}\right)}{\sqrt{36+4+9}} + 3\frac{\left(3\hat{i}-2\hat{j}+6\hat{k}\right)}{\sqrt{9+4+36}} + 1\frac{\left(2\hat{i}-3\hat{j}-6\hat{k}\right)}{\sqrt{4+9+36}}$$

$$= \frac{1}{7}\left(41\hat{i}+\hat{j}+27\hat{k}\right)$$

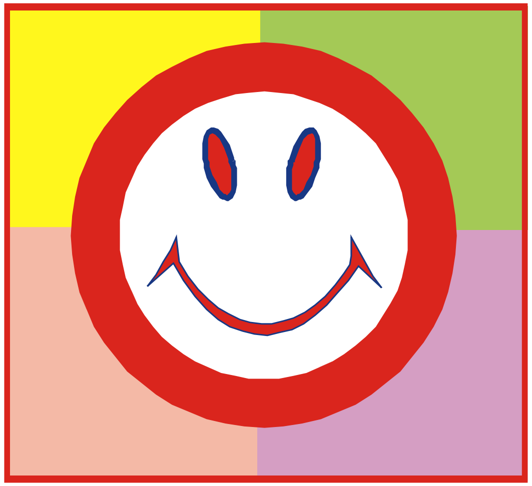
Let A and B be the points with position vectors $2\hat{i}-\hat{j}-3\hat{k}$ and $5\hat{i}-\hat{j}+\hat{k}$ respectively.

Solution Cont.

$$\vec{d} = \overrightarrow{AB} = \left(5\hat{i} - \hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} - 3\hat{k}\right) = 3\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\therefore \text{Work done} = \vec{F}.\vec{d} = \frac{1}{7}\left(41\hat{i} + \hat{j} + 27\hat{k}\right).\left(3\hat{i} + 0\hat{j} + 4\hat{k}\right)$$

$$= \frac{1}{7}(123 + 0 + 108) = \frac{231}{7} = 33 \text{ sq. units.}$$



Thank you