CHRISTIAN EMINENT COLLEGE, INDORE

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E-Content

On

" Vectors 3 "

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Mathematics

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Session

Vectors -3

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Session Objectives

- Scalar Triple Product
- Geometrical Interpretation
- Properties of Scalar Triple Product
- Vector Triple Product
- Class Exercise

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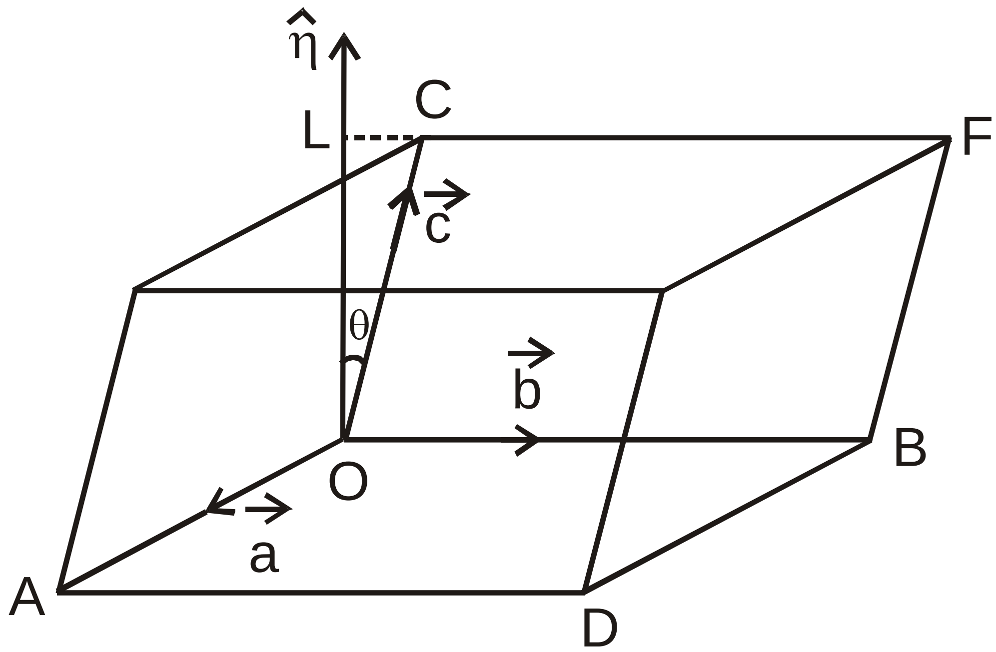
Scalar Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then, the scalar product of $\vec{a}, \vec{b}, \vec{c}$ is denoted by $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and is defined as follows:

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a}\,.\left(\vec{b}\times\vec{c}\right) = \vec{b}\,.\left(\vec{c}\times\vec{a}\right) = \vec{c}\,.\left(\vec{a}\times\vec{b}\right)$$

Geometrical Interpretation



Let OA, OB and OC be the coterminous edges of a parallelopiped such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.

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Cont.

Area of the parallelogram OADB $= \left| \vec{a} \times \vec{b} \right|$

Let θ be the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$. Let $\hat{\eta}$ be a unit vector along $\vec{a} \times \vec{b}$, then θ is also the angle between $\hat{\eta}$ and $\vec{c}$.

$$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \left(\vec{a} \times \vec{b}\right).\vec{c}$$

$$= \left(\text{Area of the parallelogram OADB}\right)\hat{\eta}\ .\ \vec{c}$$

$$= \left(\text{Area of the parallelogram OADB}\right)\left(\left|\hat{\eta}\right|\left|\vec{c}\right|\cos\theta\right)$$

$$= \left(\text{Area of the parallelogram OADB}\right)\left(\left|\vec{c}\right|\cos\theta\right) \qquad \left[\because \left|\hat{\eta}\right| = 1\right]$$

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Cont.

$$= (\text{Area of the parallelogram OADB})(\text{OL})$$

$$\left[\because \left|\vec{c}\right|\cos\theta \text{ is the projection of } \vec{c} \text{ on } \hat{\eta}\right]$$

$$= (\text{Area of the base of the parallelopiped}) \times (\text{OL})$$

$$= \text{Volume of the parallelopiped with coterminous edges } \vec{a}, \vec{b}, \vec{c}$$

Properties of Scalar Triple Product

1. For an orthonormal right-handed vector triad $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\left[\hat{i}\ \hat{j}\ \hat{k}\right] = \left(\hat{i} \times \hat{j}\right).\hat{k} = \hat{k}.\hat{k} = 1$$

2. If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted, then

$$\left(\vec{a} \times \vec{b}\right).\vec{c} = \left(\vec{b} \times \vec{c}\right).\vec{a} = \left(\vec{c} \times \vec{a}\right).\vec{b} \text{ or } \left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

3. The change of cyclic order of vectors in scalar triple product, changes the sign of the value of the scalar triple product.

i.e. $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\left[\vec{b}\ \vec{a}\ \vec{c}\right] = -\left[\vec{c}\ \vec{b}\ \vec{a}\right] = -\left[\vec{a}\ \vec{c}\ \vec{b}\right]$

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Cont.

4. In scalar triple product the position of dot and cross can be interchanged provided that the cyclic order of the vectors remains same,

   i.e. $\left(\vec{a}\times\vec{b}\right).\vec{c} = \left(\vec{b}\times\vec{c}\right).\vec{a} = \vec{a}.\left(\vec{b}\times\vec{c}\right)$

5. $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = 0.$

6. The scalar triple product of three vectoes is zero if any two of them are equal.

   i.e. $\left[\vec{a}\ \ \vec{a}\ \ \vec{b}\right] = 0$

Cont.

7. If $\vec{a}$ is parallel or collinear to $\vec{b}$, then $\vec{a} = \lambda\vec{b}$

$$\therefore \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \left[\lambda\vec{b}\ \ \vec{b}\ \ \vec{c}\right] = \lambda\left[\vec{b}\ \ \vec{b}\ \ \vec{c}\right] = \lambda.0 = 0.$$

Scalar Triple Product in Terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three vectors, then

$$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example –1

If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$, then find $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$.

$$\text{Solution}: \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= 1(6-1) + 2(-4-3) + 3(2+9)$$

$$= 5 - 14 + 33 = 24$$

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Example -2

Find the volume of the parallelopiped whose coterminous edges are represented by $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$.

Solution : We have $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 2\hat{k}$.

The volume of the parallelopiped $= \left[\vec{a}\,\vec{b}\,\vec{c}\right]$

$$= \left\| \begin{matrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 2 & -1 & 2 \end{matrix} \right\|$$

$$= \left| 2(4-1) + 3(2+2) + 4(-1-4) \right|$$

$$= |6 + 12 - 20| = |-2| = 2 \text{ cubic units.}$$

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Example –3

Show that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}, -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

Solution: Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$.

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$$= 1(15 - 12) + 2(-10 + 4) + 3(6 - 3)$$

$$= 3 - 12 + 9 = 0$$

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Example -4

Find the value of λ such that the following vectors are coplanar :

$\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = \lambda\hat{i} + \lambda\hat{j} + 5\hat{k}.$

Solution : $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0.$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ \lambda & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow 2(10 + 3\lambda) + 1(5 + 3\lambda) + 1(\lambda - 2\lambda) = 0$$

$$\Rightarrow 20 + 6\lambda + 5 + 3\lambda - \lambda = 0$$

$$\Rightarrow \lambda = -\frac{25}{8}$$

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Example –5

Show that four points whose position vectors are

$6\hat{i}-7\hat{j}, 16\hat{i}-29\hat{j}-4\hat{k}, 3\hat{j}-6\hat{k}$ and $2\hat{i}+5\hat{j}+10\hat{k}$ are coplanar.

Solution: Let A, B, C, D be the given points.

The given points will be coplanar $\Leftrightarrow \left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0$

$$\overrightarrow{AB}=\left(16\hat{i}-29\hat{j}-4\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)=10\hat{i}-22\hat{j}-4\hat{k}$$

$$\overrightarrow{AC}=\left(3\hat{j}-6\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)=-6\hat{i}+10\hat{j}-6\hat{k}$$

$$\overrightarrow{AD}=\left(2\hat{i}+5\hat{j}+10\hat{k}\right)-\left(6\hat{i}-7\hat{j}\right)=-4\hat{i}+12\hat{j}+10\hat{k}$$

Solution Cont.

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = \begin{vmatrix} 10 & -22 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix}$$

$$= 10(100+72) + 22(-60-24) - 4(-72+40)$$

$$= 1720 - 1848 + 128 = 0$$

Example –6

Prove that $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$.

Solution: $\text{LHS}=\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]$

$$=\left(\vec{a}+\vec{b}\right).\left\{\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right\}$$

$$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right)$$

$$=\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\right)$$

Solution Cont.

$$=\vec{a}.\left(\vec{b}\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left[\vec{a}\;\vec{b}\;\vec{c}\right]+\left[\vec{a}\;\vec{b}\;\vec{a}\right]+\left[\vec{a}\;\vec{c}\;\vec{a}\right]+\left[\vec{b}\;\vec{b}\;\vec{c}\right]+\left[\vec{b}\;\vec{b}\;\vec{a}\right]+\left[\vec{b}\;\vec{c}\;\vec{a}\right]$$

$$=\left[\vec{a}\;\vec{b}\;\vec{c}\right]+0+0+0+0+\left[\vec{a}\;\vec{b}\;\vec{c}\right]$$

$$=2\left[\vec{a}\;\vec{b}\;\vec{c}\right]=RHS$$

Vector Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors, then the vectors $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called the vector triple product of $\vec{a}, \vec{b}, \vec{c}$.

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

and $$(\vec{a}\times\vec{b})\times\vec{c} = -\{\vec{c}\times(\vec{a}\times\vec{b})\} = -\{(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}\} = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}$$

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Example –7

Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}.$

Solution: $\text{LHS}=\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})$

$$=\left\{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}\right\}+\left\{(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}\right\}+\left\{(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}\right\}$$

$$=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{b}.\vec{c})\vec{a}-(\vec{a}.\vec{c})\vec{b}$$

$$=\vec{0}=\text{RHS}$$

Example –8

For any vector $\vec{a}$, prove that $\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)=2\vec{a}$.

Solution: $\text{LHS}=\hat{i}\times\left(\vec{a}\times\hat{i}\right)+\hat{j}\times\left(\vec{a}\times\hat{j}\right)+\hat{k}\times\left(\vec{a}\times\hat{k}\right)$

$$=\left(\hat{i}\,.\,\hat{i}\right)\vec{a}-\left(\hat{i}\,.\,\vec{a}\right)\hat{i}+\left(\hat{j}\,.\,\hat{j}\right)\vec{a}-\left(\hat{j}\,.\,\vec{a}\right)\hat{j}+\left(\hat{k}\,.\,\hat{k}\right)\vec{a}-\left(\hat{k}\,.\,\vec{a}\right)\hat{k}$$

$$=\vec{a}-\left(\hat{i}\,.\,\vec{a}\right)\hat{i}+\vec{a}-\left(\hat{j}\,.\,\vec{a}\right)\hat{j}+\vec{a}-\left(\hat{k}\,.\,\vec{a}\right)\hat{k}$$

$$=3\vec{a}-\left\{\left(\hat{i}\,.\,\vec{a}\right)\hat{i}+\left(\hat{j}\,.\,\vec{a}\right)\hat{j}+\left(\hat{k}\,.\,\vec{a}\right)\hat{k}\right\}$$

$=3\vec{a}-\vec{a}$ $\quad\left[\text{If } \vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \text{ then } (\hat{i}.\vec{a})\hat{i}+(\hat{j}.\vec{a})\hat{j}+(\hat{k}.\vec{a})\hat{k}=\vec{a}\right]$

$$=2\vec{a}=\text{RHS}$$

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Example –9

Prove that $\left[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\right]=\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2$.

Solution: $\text{LHS}=\left[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\right]$

$$=\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\vec{d}\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right),\ \text{where } \vec{d}=\vec{a}\times\vec{b}$$

$$=\left\{\left(\vec{d}\ .\ \vec{c}\right)\vec{b}-\left(\vec{d}\ .\ \vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\left(\left(\vec{a}\times\vec{b}\right).\vec{c}\right)\vec{b}-\left(\left(\vec{a}\times\vec{b}\right).\vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$$

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Solution Cont.

$$=\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}-\left[\vec{a}\ \vec{b}\ \vec{b}\right]\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}-(0)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}\right\}.\left(\vec{c}\times\vec{a}\right)$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left(\vec{b}.\left(\vec{c}\times\vec{a}\right)\right)$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2=\text{RHS}$$

Example –10

If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$

show that $\left(\vec{a} \times \vec{b}\right) \times \vec{c} \neq \vec{a} \times \left(\vec{b} \times \vec{c}\right)$.

Solution :

$$\text{LHS} = \left(\vec{a} \times \vec{b}\right) \times \vec{c} = -\left\{\vec{c} \times \left(\vec{a} \times \vec{b}\right)\right\} = -\left\{\left(\vec{c}.\vec{b}\right)\vec{a} - \left(\vec{c}.\vec{a}\right)\vec{b}\right\} = \left(\vec{c}.\vec{a}\right)\vec{b} - \left(\vec{c}.\vec{b}\right)\vec{a}$$

$$= \left\{\left(\hat{i} + \hat{j} - 2\hat{k}\right).\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\right\}\left(2\hat{i} - \hat{j} + \hat{k}\right)$$

$$- \left\{\left(\hat{i} + \hat{j} - 2\hat{k}\right).\left(2\hat{i} - \hat{j} + \hat{k}\right)\right\}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= (1 + 2 - 6)\left(2\hat{i} - \hat{j} + \hat{k}\right) - (2 - 1 - 2)\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

Solution Cont.

$$= \left(-6\hat{i} + 3\hat{j} - 3\hat{k}\right) + \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= -5\hat{i} + 5\hat{j}$$

$$RHS = \vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$$

$$= \left\{\left(\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(\hat{i} + \hat{j} - 2\hat{k}\right)\right\}\left(2\hat{i} - \hat{j} + \hat{k}\right)$$

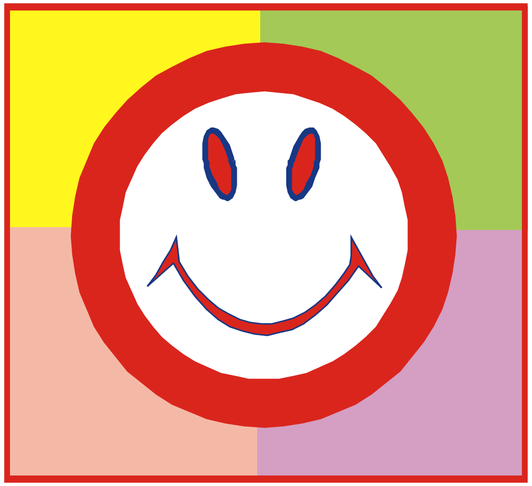
$$- \left\{\left(\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(2\hat{i} - \hat{j} + \hat{k}\right)\right\}\left(\hat{i} + \hat{j} - 2\hat{k}\right)$$

Solution Cont.

$$=(1+2-6)\left(2\hat{i}-\hat{j}+\hat{k}\right)-(2-2+3)\left(\hat{i}+\hat{j}-2\hat{k}\right)$$

$$=\left(-6\hat{i}+3\hat{j}-3\hat{k}\right)-\left(3\hat{i}+3\hat{j}-6\hat{k}\right)$$

$$=-9\hat{i}+3\hat{k}\neq LHS$$



Thank you