CHRISTIAN EMINENT COLLEGE, INDORE

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E-Content

On

" Vectors 2 "

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Mathematics

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Session

Vectors -2

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Session Objectives

- Vector (or Cross) Product
- Geometrical Representation
- **Properties of Vector Product**
- **Vector Product in Terms of Components**
- **Applications: Vector Moment of a Force about a Point, about a Line**
- Class Exercise

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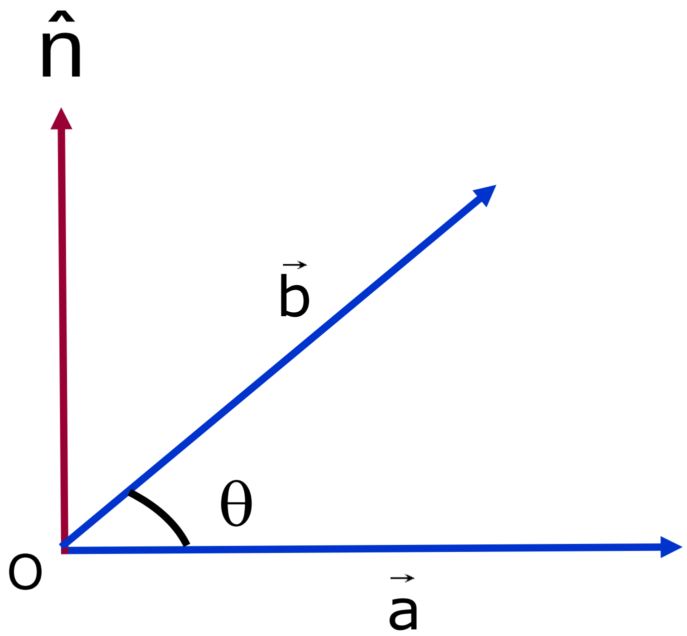
Vector (or Cross) Product



Let $\vec{a}, \vec{b}$ be two non - zero non - parallel vectors. Then,

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

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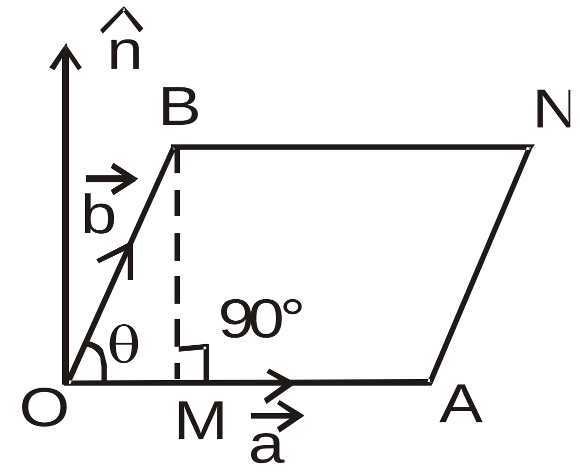
Note

1. If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$, then θ is not defined as $\vec{0}$ has no direction and $\hat{n}$ is not defined. In this case $\vec{a} \times \vec{b} = 0$.

2. If $\vec{a}$ and $\vec{b}$ are collinear i.e. if $\theta = 0$ or π, then $\vec{a} \times \vec{b} = 0$.

Geometrical Representation



Area of parallelogram OANB

$$= OA \,.\, BM = OA \,.\, OB \sin\theta$$

$$= \left|\overrightarrow{OA}\right| . \left|\overrightarrow{OB}\right| \sin\theta$$

$$= \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta$$

$$= \left|\vec{a} \times \vec{b}\right|$$

Also, area of $\Delta OAB = \frac{1}{2}$ area of parallelogram OANB $= \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

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Properties of Vector Product

1. Vector product is not commutative

i.e. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

In fact $\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$

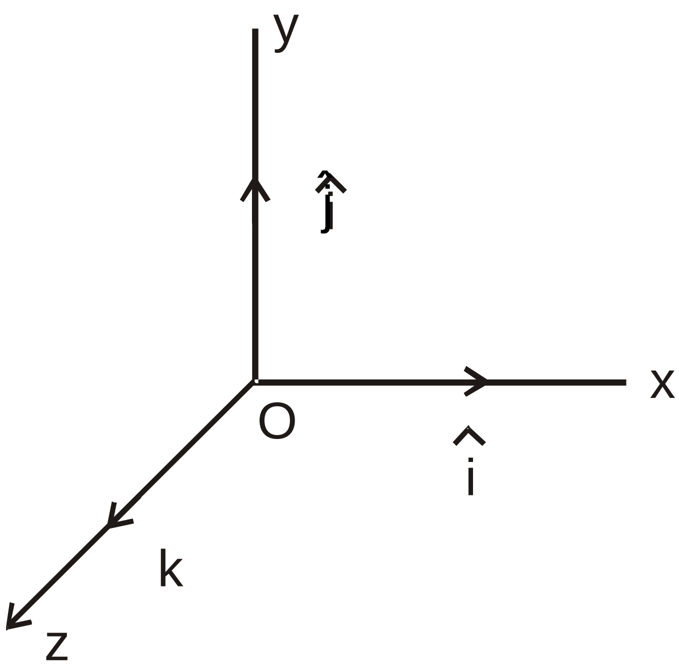
2. Vector product is distributive over vector addition

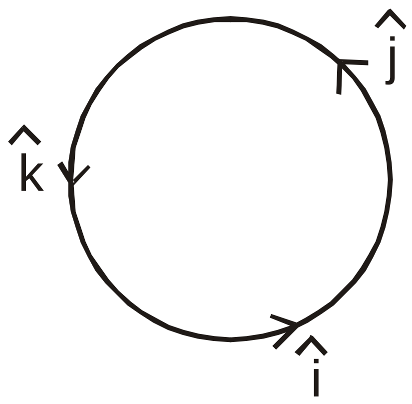
$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ [Left distributive law]

$(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$ [Right distributive law]

Properties of Vector Product Cont.

3. Vector product of orthogonal traid of unit vectors $\hat{i}, \hat{j}, \hat{k}$





Properties of Vector Product Cont.

$$\hat{i} \times \hat{i} = |\hat{i}||\hat{i}| \sin 0° \, \hat{n} = \vec{0}$$

Similarly, $\hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

$$\hat{i} \times \hat{j} = \hat{k}, \ \ \hat{j} \times \hat{k} = \hat{i}, \ \ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \ \ \hat{k} \times \hat{j} = -\hat{i}, \ \ \hat{i} \times \hat{k} = -\hat{j}$$

4. If $\vec{a}$ and $\vec{b}$ are two vectors and m is real number, then

$$\left(m\vec{a}\right) \times \vec{b} = m\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(m\vec{b}\right)$$

Vector Product in Terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \times \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Vectors Normal to the Plane of Two Given Vectors

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n} \Rightarrow \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}| \sin\theta}$$

$\Rightarrow \hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

$-\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

Vector of magnitude 'λ' normal to the plane of $\vec{a}$ and $\vec{b}$ is given by

$$\pm \frac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

Lagrange's Identity

If $\vec{a}$ and $\vec{b}$ are any two vectors, then

$$\left|\vec{a}\times\vec{b}\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - \left(\vec{a}.\vec{b}\right)^2 \text{ or } \left|\vec{a}\times\vec{b}\right|^2 + \left(\vec{a}.\vec{b}\right)^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2$$

Example –1

Find a unit vector perpendicular to the plane containing

the vectors $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

Solution: We have $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = (1-2)\hat{i} - (2-1)\hat{j} + (4-1)\hat{k}$$

$$= -\hat{i} - \hat{j} + 3\hat{k}$$

Solution Cont.

$\therefore$ A unit vector perpedicular to the plane containing $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|} = \pm \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{(-1)^2 + (-1)^2 + (3)^2}}$$

$$= \pm \frac{-\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}}$$

Example –2

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the non-collinear points A, B, C respectively in space, show that $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ is perpendicular to the plane ABC.

Solution: The vector perpendicular to the plane ABC is $\overrightarrow{AB}\times\overrightarrow{AC}$.

$$\overrightarrow{AB}=\vec{b}-\vec{a} \text{ and } \overrightarrow{AC}=\vec{c}-\vec{a}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)$$

$$=\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}$$

$$=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$$

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Example –3

If $\left|\vec{a}\right| = \sqrt{26}$, $\left|\vec{b}\right| = 7$ and $\left|\vec{a}\times\vec{b}\right| = 35$, then find $\vec{a}\,.\,\vec{b}$. (CBSE 2002)

Solution : We have $\left|\vec{a}\times\vec{b}\right|^2 + \left(\vec{a}\,.\,\vec{b}\right)^2$

$$= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \sin^2\theta + \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \cos^2\theta$$

$$\Rightarrow \left|\vec{a}\times\vec{b}\right|^2 + \left(\vec{a}\,.\,\vec{b}\right)^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$$

$$\Rightarrow 35^2 + \left(\vec{a}\,.\,\vec{b}\right)^2 = 26\times 7^2$$

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Solution Cont.

$$\Rightarrow \left(\vec{a}\,.\,\vec{b}\right)^2 = 26 \times 7^2 - 35^2 = 49$$

$$\Rightarrow \vec{a}\,.\,\vec{b} = 7$$

Example –4

If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, then find $\vec{a} \times \vec{b}$.

Verify that $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

Solution: We have $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= (10 - 9)\hat{i} - (-5 - 6)\hat{j} + (3 + 4)\hat{k}$$

$$= \hat{i} + 11\hat{j} + 7\hat{k}$$

Solution Cont.

$\vec{a}$ and $\left(\vec{a}\times\vec{b}\right)$ are perpendicular to each other if

$$\vec{a}\,.\left(\vec{a}\times\vec{b}\right)=0$$

$$\vec{a}\,.\left(\vec{a}\times\vec{b}\right)=\left(\hat{i}-2\hat{j}+3\hat{k}\right).\left(\hat{i}+11\hat{j}+7\hat{k}\right)$$

$$=1-22+21=0$$

Example –5

If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$,
$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$, then show that $\vec{b} = \vec{c}$. (CBSE 1997C)

Solution: $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a} \neq \vec{0}$

$$\Rightarrow \vec{a}.\vec{b} - \vec{a}.\vec{c} = 0 \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \vec{a}.\left(\vec{b} - \vec{c}\right) = 0 \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \left(\vec{b} - \vec{c}\right) = 0 \text{ or } \vec{a} \perp \left(\vec{b} - \vec{c}\right)$$

$$\Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \perp \left(\vec{b} - \vec{c}\right) \qquad \ldots(i)$$

Solution Cont.

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0 \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \vec{a} \times \left(\vec{b} - \vec{c}\right) = 0 \text{ and } \vec{a} \neq \vec{0}$$

$$\Rightarrow \left(\vec{b} - \vec{c}\right) = 0 \text{ or } \vec{a} \,||\, \left(\vec{b} - \vec{c}\right)$$

$$\Rightarrow \vec{b} = \vec{c} \text{ or } \vec{a} \,||\, \left(\vec{b} - \vec{c}\right) \qquad \ldots(ii)$$

From (i) and (ii), $\vec{b} = \vec{c}$

Applications

1. Area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b} = \left|\vec{a} \times \vec{b}\right|$

2. Area of triangle with adjacent sides $\vec{a}$ and $\vec{b} = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

3. Area of parallelogram with diagonals $\overrightarrow{d_1}$ and $\overrightarrow{d_2} = \frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

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Example -6

Find the area of a parallelogram determined by the vectors

$\hat{i} + \hat{j} + \hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Solution: Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$, where $\vec{a}$ and $\vec{b}$ are adjacent sides of the parallelogram.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = (1+2)\hat{i} - (1-3)\hat{j} + (-2-3)\hat{k}$$

$$= 3\hat{i} + 2\hat{j} - 5\hat{k}$$

Solution Cont.

$$\text{Area of parallelogram} = \left|\vec{a} \times \vec{b}\right|$$

$$= \sqrt{3^2 + 2^2 + (-5)^2} = \sqrt{38}\ \text{sq. units.}$$

Example -7

Find the area of the triangle formed by the points A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

$$\text{Solution}: \overrightarrow{AB} = \left(2\hat{i} + 3\hat{j} + 5\hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right) = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{AC} = \left(\hat{i} + 5\hat{j} + 5\hat{k}\right) - \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 0\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = (6 - 12)\hat{i} - (3 - 0)\hat{j} + (4 - 0)\hat{k}$$

$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

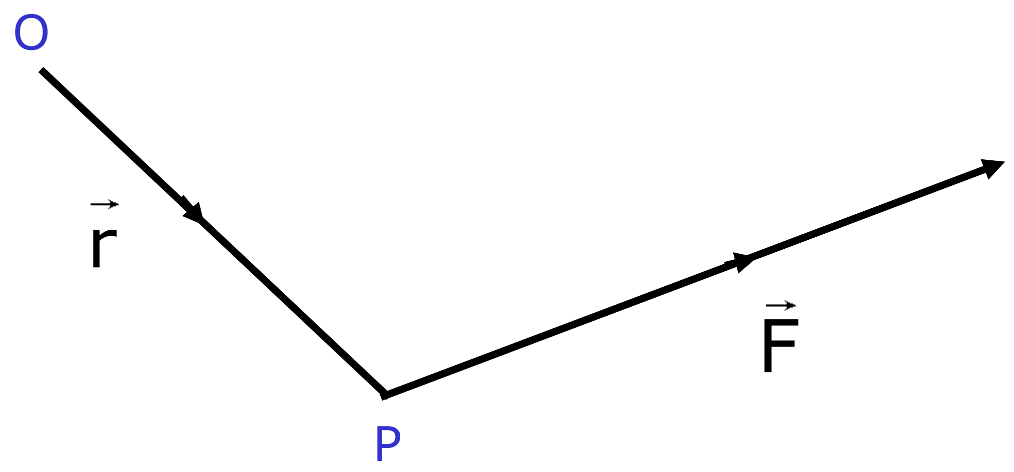
Solution Cont.

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\sqrt{(-6)^2 + (-3)^2 + (4)^2}$$

$$= \frac{\sqrt{61}}{2} \text{ sq. units}$$

Moment of Force About a Point



Let $\vec{r}$ be the position vector of P relative to O. Then the moment (or torque) of $\vec{F}$ about the point O is

$$\vec{M} = \vec{r} \times \vec{F}$$

Example -8

Find the moment of a force represented by $\hat{i}+\hat{j}+\hat{k}$ acting through the point B with position vector $-2\hat{i}+3\hat{j}+\hat{k}$, about the point A with position vector $\hat{i}+2\hat{j}+3\hat{k}$.

Solution : We have $\vec{F}=\hat{i}+\hat{j}+\hat{k}$

$$\vec{r}=\overrightarrow{AB}=\left(-2\hat{i}+3\hat{j}+\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=-3\hat{i}+\hat{j}-2\hat{k}$$

Solution Cont.

The moment of the force $\vec{F}$ acting through B about the point A is given by

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (1+2)\hat{i} - (-3+2)\hat{j} + (-3-1)\hat{k}$$

$$= 3\hat{i} + \hat{j} - 4\hat{k}$$

Moment of Force About a Line

The moment of $\vec{F}$ about a line L is

$$\left(\vec{r} \times \vec{F}\right).\hat{a}$$

where $\hat{a}$ is a unit vector in the direction of the line L, and

$\overrightarrow{OP} = \vec{r}$ where O is any point on the line.

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Example -9

Let $\vec{F} = 2\hat{i} - 4\hat{j} - 3\hat{k}$ acts at the point P with position vector $\hat{i} - 2\hat{j} + 3\hat{k}$. Find the moment of $\vec{F}$ about the line through the origin O in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

$$\text{Solution}: \overrightarrow{OP} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -4 & -3 \end{vmatrix}$$

$$= (6+12)\hat{i} - (-3-6)\hat{j} + (-4+4)\hat{k}$$

$$= 18\hat{i} + 9\hat{j}$$

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Solution Cont.

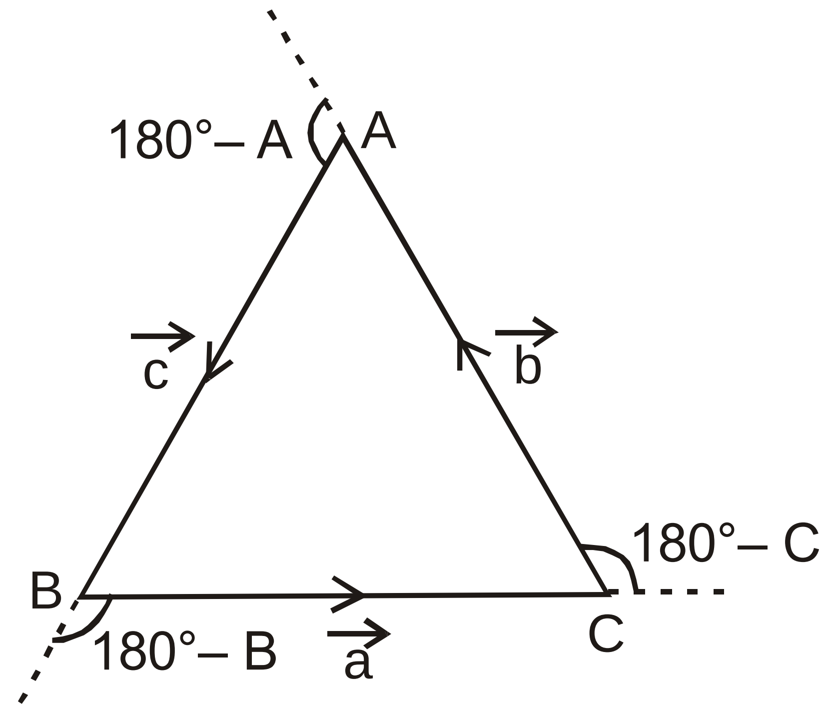
The moment of the force $\vec{F}$ about the given line is

$$\left(\overrightarrow{OP}\times\vec{F}\right).\left(\frac{\vec{a}}{\left|\vec{a}\right|}\right)=\left(18\hat{i}+9\hat{j}\right).\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)=\frac{18+9}{\sqrt{3}}=9\sqrt{3}$$

Geometrical Problem
Example -10

In a triangle ABC, prove by vector method that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Solution:

By triangle law of vector addition

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

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Solution Cont.

$$\Rightarrow \vec{a}\times\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{a}\times\vec{0}$$

$$\Rightarrow \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{0}\Rightarrow \vec{a}\times\vec{b}=-\vec{a}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a} \qquad \ldots(i)$$

$$\vec{b}\times\left(\vec{a}+\vec{b}+\vec{c}\right)=\vec{b}\times\vec{0}$$

$$\Rightarrow \vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}\Rightarrow \vec{b}\times\vec{c}=-\vec{b}\times\vec{a}$$

$$\Rightarrow \vec{b}\times\vec{c}=\vec{a}\times\vec{b} \qquad \ldots(ii)$$

Solution Cont.

From (i) and (ii), we get

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

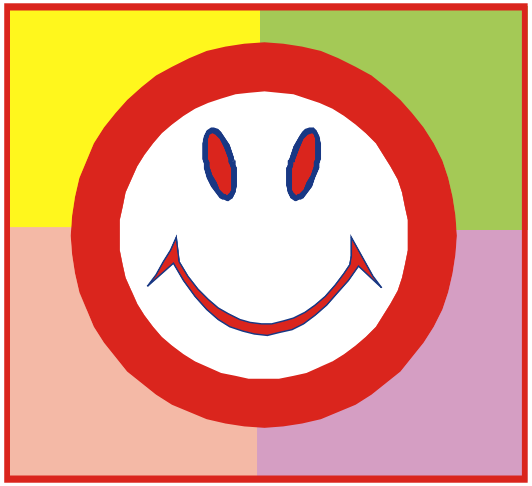
$$\Rightarrow \left|\vec{a}\times\vec{b}\right|=\left|\vec{b}\times\vec{c}\right|=\left|\vec{c}\times\vec{a}\right|$$

$$\Rightarrow ab\sin(180°-C)=bc\sin(180°-A)=ac\sin(180°-B)$$

$$\Rightarrow ab\sin C=bc\sin A=ac\sin B$$

$$\Rightarrow \frac{\sin C}{c}=\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\Rightarrow \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$



Thank you