CHRISTIAN EMINENT COLLEGE, INDORE

B.SC.-I Year (Mathematics)

CONE

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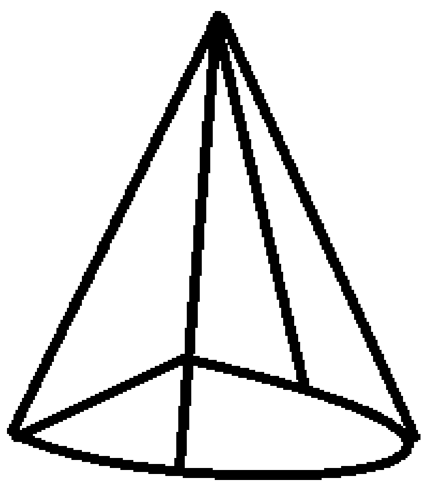
Definition- A Cone is a surface generated by a moving straight line which passes through a fixed point and intersect a given curve or a touches a given surface.

Vertex

Generator Line



Guiding Curve

Theorem: General equation of the cone with vertex at origin is homogenous of second degree of the type $\boldsymbol{ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0}$. D. R's must satisfy eqn. of cone.

Proof: Given generator is If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=$ r

∴ any point on the generator is (Ir, mr, nr), which lies on the cone

$$ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0-(1)$$

⇒ This point (Ir, mr, nr) must satisfy (1)

∴ we havea $(Ir)^2+b(mr)^2+c(nr)^2+2h(mr)(nr)+2g(nr)(mr)+2f(mr)(Ir)=0$

i.e

$$r^2(a(I)^2 + b(m)^2 + c(n)^2 + 2hlm + 2g\ln + 2fmn) = 0 \quad But r^2 \neq 0,$$

$$\therefore al^2 + bm^2 + c^2 + 2hlm + 2gln + 2fmn = 0$$

i.e

D.R's satisfy the eqn. of cone.

General equation of the cone with vertex at origin and passing through coordinate axis is $hxy + gzx + fyz = 0$.

Proof: Let General equation of the cone with vertex at origin be $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0. - - (1)$

If (1) passes through coordinate axis (they areas generators) then
D.R.'s of x-axis, y-axis and z-axis must satisfy eqn. (1) by cor.(1)
But D.R's of x-axis are 1,0,0, they satisfy eqn. (1) => $a(1)^1 + 0 + 0 + 0 + 0 + 0 = 0, \Rightarrow a = 0$
Similarly D.R's of y-axis are 0,1,0 and z-axis 0,0,1 must satisfy (1)
$\Rightarrow b = 0$ and $c = 0$
$\therefore$ eqn. (1) becomes $0 + 0 + 0 + 2hxy + 2gzx + 2fyz = 0.$
i.e hxy +gzx +fyz = 0
Thus equation of the cone with vertex at origin and passing through coordinate axis
hxy +gzx +fyz=0.
Examples:

Prove that the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, where $l^2 + 2m^2 - 3n^2 = 0$ is a generator for the cone $x^2 + 2y^2 - 3z^2 = 0$

Proof: We know that if $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator for the cone $x^2 + 2y^2 - 3z^2 = 0$ then by cor. (1),

D. $R's$l, m, n of the generator must satisfy the eqn of cone.
∴ we have $l^2 + 2m^2 - 3n^2 = 0$ which is true.

Find eqn. of cone generated by the line through $(1, 2, 3)$ whose D.C.'s satisfy the eqn. $21^2 + 3m^2 - 4n^2 = 0$

Soln.: Eqn of generator passing through the point (1,2,3) with D.C's I, m, n is

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = \mathbf{r} \text{ (say)}$$

$\Rightarrow \mathrm{I} = \dfrac{x-1}{r}, \mathrm{m} = \dfrac{y-2}{r}, \mathrm{n} = \dfrac{z-3}{r}$

By cor. (1) we know that D. C's of generator satisfy the eqn. of cone and that eqn. is given by $2|^2 + 3m^2 - 4n^2 = 0 \cdots - (1)$

Substitute I, m, n in (1) we get $2\left(\frac{x-1}{r}\right)^2 + 3\left(\frac{y-2}{r}\right)^2 - 4\left(\frac{z-3}{r}\right)^2 = 0$

i.e $2(x-1)^2 + 3(y-2)^2 - 4(z-3)^2 = 0$.

i.e $2x^2 + 3y^2 - 4z^2 - 4x - 12y + 24z - 22 = 0$ which is required eqn. of cone.

Find eqn. to the cone with vertex at $(0, 0, 0)$ which passes through the curve of intersection of plane $1x + my + nz = p$ and $ax^2 + by^2 + cz^2 = 1$.

Soln.: Guiding curve is intersection of $|\ x + my + nz = p \cdots - (1)$ and $ax^2 + by^2 + cz^2 = 1, \cdots (2)$

Homogenizing (1) and (2) we get, $|\ x + my + nz = pt \quad -$ (3) ($b'cz$ this eqn. of degree 1) $ax^2 + by^2 + cz^2 = t^2 \cdots (4)$ ($b'cz$ this eqn. of degree 2) Substitute $t = \frac{lx+my+nz}{p}$ from (3) in (4) we get $ax^2 + by^2 + cz^2 = \left(\frac{lx+my+nz}{p}\right)^2$ i.e $p^2(ax^2 + by^2 + cz^2) = (lx + my + nz)^2$, which is req. eqn. of cone

Theorem: General equation of the cone with vertex at origin is homogenous of second degree of the type $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0$. D. R's must satisfy eqn. of cone.

Proof: Given generator is If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = \mathrm{r}$

∴ any point on the generator is (Ir, mr, nr), which lies on the cone

$ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0 - (1) \Rightarrow$ This point (Ir, mr, nr) must satisfy (1) $\quad\therefore$ we havea $(Ir)^2 + b(mr)^2 + c(nr)^2 + 2h(mr)(nr) + 2g(nr)(mr) + 2f(mr)(Ir) = 0$ i.e $r^2(a(I)^2 + b(m)^2 + c(n)^2 + 2hlm + 2g\ln + 2fmn) = 0$ $\quad Butr^2 \neq 0, \therefore al^2 + bm^2 + c^2 + 2hlm + 2gln + 2fmn = 0$ i.e D.R's satisfy the eqn. of cone Cor.2: General equation of the cone with vertex at origin and passing through coordinate axis is $hxy + gzx + fyz = 0$.

Another Proof: Let General equation of the cone with vertex at origin be $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0. - -$ (1) If (1) passes through coordinate axis (they areas generators) then D.R.'s of x-axis, y-axis and z-axis must satisfy eqn. (1) by cor.(1)
But D.R's of x-axis are 1,0,0, they satisfy eqn. (1) => $a(1)^1 + 0 + 0 + 0 + 0 + 0 = 0, \Rightarrow a = 0$
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By cor. (1) we know that D. C's of generator satisfy the eqn. of cone and that eqn. is given by $2|^2 + 3m^2 - 4n^2 = 0 \cdots - (1)$

Substitute I, m, n in (1) we get $2\left(\frac{x-1}{r}\right)^2 + 3\left(\frac{y-2}{r}\right)^2 - 4\left(\frac{z-3}{r}\right)^2 = 0$

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Homogenizing (1) and (2) we get, $\mathrm{l}\, x + my + nz = pt$ $-$ (3) ($b'cz$ this eqn. of degree 1) $ax^2 + by^2 + cz^2 = t^2 \cdots (4)$ ($b'cz$ this eqn. of degree 2) Substitute $\mathrm{t} = \frac{lx+my+nz}{p}$ from (3) in (4) we get $\mathrm{ax}^2 + \mathrm{by}^2 + \mathrm{cz}^2 = \left(\frac{lx+my+nz}{p}\right)^2$ i.e $\mathrm{p}^2(\mathrm{ax}^2 + \mathrm{by}^2 + \mathrm{cz}^2) = (lx + my + nz)^2$, which is req. eqn. of cone

Prove that eqn. $2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0$ represents cone & find its vertex.

Soln.: Given eqn. be $f(x, y, z) = 2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0 \ldots - (1)$

Make it homogeneous of second degree by introducing t

i.e. $F(x, y, z, t) = 2x^2 - 8xy - 4yz - 4xt - 2yt + 6zt + 35t^2 = 0 \cdots - (2)$

Differentiate (2) partially w.r.t x, y, z, and t,

Next, $\frac{\partial F}{\partial y} = 0 \Rightarrow -8\mathrm{x} - 4\mathrm{z} - 2\mathrm{t} = 0$, i.e

$\dfrac{\partial F}{\partial z} = 0 \Rightarrow -4y + 6 = 0, \Rightarrow y = \dfrac{3}{2}$

$8x + 4z + 2 = 0 \cdots - (4)$ by sub. $t = 1$

$\frac{\partial F}{\partial t} = 0 \Rightarrow -4\mathrm{x} - 2\mathrm{y} + 6\mathrm{z} + 70\mathrm{t} = 0$ i.e

$-4x - 2y + 6z + 70 = 0 \ldots - (6), t = 1$

Put $y = \frac{3}{2}$ in (3) we get $4x - 8\left(\frac{3}{2}\right) - 4 = 0, \Rightarrow x = 4$
Put $x = 4$ in (5) we get, $32 + 4z + 2 = 0 \Rightarrow 4z = -34 \quad \therefore z = -\frac{17}{2}$
Put $x = 4, y = \frac{3}{2}$ and $z = \frac{-17}{2}$ in (6) we get
$-16 - 2\left(\frac{3}{2}\right) + 6\left(-\frac{17}{2}\right) + 70 = -16 - 3 - 51 + 70 = -70 + 70 = 0 \therefore$
(6) is satisfied by $\left(4, \frac{3}{2}, -\frac{17}{2}\right)$
$\Rightarrow$ Eqn. (1) represents cone with $\left(4, \frac{3}{2}, -\frac{17}{2}\right)$ as vertex.

Find the eqn. of the right circular cone with vertex $(0,0,0)$, eqn. of the axis as $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and semi vertical angle θ is $60°$.
Soln: Let $P(x, y, z)$ be any point on the surface of the cone.
Squaring both the sides we get ,
D.R's of axis are 1,2,3 and $\theta = 60° \therefore \cos\ 60 = \frac{x1 + y2 + z3}{\sqrt{x^2+y^2+z^2}\sqrt{1+2^2+3^2}}$
i.e $\frac{1}{2} = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{14}}$

$14(x^2 + y^2 + z^2) = 4(x + 2y + 3z)^2$
i.e $7(x^2 + y^2 + z^2) = 2(x + 2y + 3z)^2$, on simplifying we get,
$5x^2 - y^2 - 11z^2 - 8xy - 24yz - 12zx = 0$, which is req. eqn. of rt. circular cone

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